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Logic for Mathematical Writing

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Abstract

In the School of Mathematical Sciences at Queen Mary in the University of London we have been running a module that teaches the students to write good mathematical English. The module is for second-year undergraduates and has been running for three years. It is based on logic, but the logic—though mathematically precise—is informal and doesn't use logical symbols. Some theory of definitions is taught in order to give a structure for mathematical descriptions, and some natural deduction rules form a basis for writing mathematical arguments. Alongside this logical material, the students have weekly exercises that involve writing informal explanations of simple mathematical ideas for non-mathematicians.

Keywords: writing, mathematical exposition, logic, informal logic, definition, discharging assumptions, natural deduction, Pascal.

We describe a way of using logic to give structure and motivation to a course teaching mathematical students to write English. This use of logic is—to the best of our knowledge—entirely new. Apart from our own preliminary experiments it is still untested. But positive reactions from students and colleagues encourage us to report our progress with this course.

Our warm thanks to the organisers and editors of the Salamanca meeting, and also to the audience and referees for their friendly criticisms.

1 Teaching writing

At Queen Mary some five years ago, the command came down from on high to all teaching departments:

Teach your undergraduate students to write English.

In the Mathematics department we were bemused at first. We knew our students had problems with writing. One of our teaching staff (Franco Vivaldi) had been fighting a lonely battle to make his students write correct and meaningful sentences. But nobody saw a way to bring writing into the Mathematics curriculum. Many of the lecturers felt that in any case they had no particular competence to teach English writing. For several months we ignored the college's instruction. But the college was very persistent; it even hired a coordinator to persuade all departments to accept their responsibilities for teaching writing, and there were hints of money to support any suitable initiative. (Later the money materialised and funded

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Edmund as a teaching assistant for the course.) We were made aware that other universities in Britain and North America had schemes for teaching their students to write. Undeniably there is a need for scientists and mathematicians to be able to communicate with a more general audience—for example in exploring complex issues such as global warming and genetic engineering. Even scientifically established areas such as evolution have come into public question. Though increasing the communication component in a mathematics degree can't hope to address these wider issues, it is an important step forward.

In the end, therefore, we thought seriously about the matter. Skills like communication are best taught in the process of learning something else. As a result, the existing courses that address this need have big differences both in approach and focus. (In Britain the mathematical courses that require the students to write English essays are largely in history of mathematics. We were also aware of a course on the role of mathematics in society, and a course on the nature of mathematical thinking.) An additional complication was that few reports of these courses existed, even at an anecdotal level. This paper starts to address this hole in the literature, discussing the methods that we used and the effects that we saw.

2 Mathematical writing exercises

The initial requirement was obvious. We needed a stock of writing exercises that involved mathematics. An obvious format was to explain mathematical puzzles to non-mathematicians. Here are two of the many examples that we devised. In retrospect these were two of the more successful ones, because they had minimal mathematical prerequisites and they seemed to grasp the imagination of the students.

- A popular scientific journal has invited you to write a short column answering questions sent in by readers. Write an article on the following question. Your article should be at most 250 words; it can be shorter.

  > When we look at ourselves in a mirror, why do we see ourselves with right and left reversed, but not upside down?

- Librarians pride themselves on being able to give helpful answers to any questions put to them. You are the leader of a team of librarians in the college library. Write a memo to the team, giving sound advice on how to respond to the following questions from library users. You should expose the reasons why people get the wrong answer.

(a) Why is it that when the price of petrol goes up 10% and then comes down 10%, it doesn't finish up where it started?
(b) I drive ten miles at 30 miles an hour, and then another ten miles at 50 miles an hour. It seems to me my average speed over the journey should be 40 miles an hour, but it doesn't work out that way. Why not?

Note that the examples above spell out who the students are writing for. We emphasised from the start that all writing is for some intended audience, and different audiences need different styles of writing. In fact some Mathematics lecturers had insisted that there was no need for special tuition in writing, because writing out lecture notes and solutions to exercises would give the students practice in writing mathematics. They missed an important point: lecture notes and exercise solutions are only intended to be read by mathematicians, so they give no practice in the important art of explaining mathematical ideas clearly to non-mathematicians.

Mathematicians have a reputation for being bad at putting themselves in other people's minds. The question about librarians was written to force the students to think about two other kinds of mind: the librarians and the people they have to advise. To compensate for this social subtlety, the mathematics involved was very easy. Another exercise that we used for the same purpose was the Monty Hall paradox (see Ian Stewart [6] p. 97ff); we asked the students not just to solve it, but to try to get into the minds of the many people who are convinced by the wrong solution, so as to show them where they went wrong. This was a popular exercise—in fact several of the students chose it for their oral presentation later in the course. But not all of the students understood enough probability theory to handle it convincingly.

The course ran through a twelve-week term (in practice eleven weeks), and in each week the students were required to write an essay of the sort illustrated above. We marked each exercise promptly, and handed them back to the students in a weekly discussion session. In the session the students were split into groups of about four people. The students in each group were required to read each other's answers to the exercises, and to write down comments on the answers and on our marking of them. Some students took these discussion sessions very seriously and certainly benefited from them. Others didn't, and so we introduced an incentive: initially we marked only out of 90, leaving it to the discussion groups to add further marks out of 10. We also raised some of our initial marks on the basis of the discussion groups' comments.

This was one area in which we could gain a good, if subjective, view of student development. In the early stages many students were lost when attempting to assess someone else's work, beyond a vague comment on whether it was good or bad (not always in agreement with us). But as the course progressed, their ability to analyse did seem to grow. In part this was because they learned a vocabulary for describing features of writing. (We offered them labels for certain kinds of poor writing, like "Henry James sentence" for long and convoluted sentences.) However, as they were forced to form an opinion and justify it each week their other opinions also came into sharper focus. In many cases they could be more critical in their marking than we were.

We allowed the students to resubmit writing exercises for extra marks, after the first marking and the discussion group comments. The results of this were a little disappointing. Some students resubmitted with only minimal changes, in a rush just before the deadline at the end of the term; their essays were very little improved and they got few marks added. But there were always some students who took the opportunity to rewrite soon after the first writing, while things were still fresh in their minds; their essays were usually much better the second time round, and they could see that they had learned something.

The puzzle-explanation side of the course worked well. But we were aware from the beginning that by itself it would never make a convincing module alongside the other mathematics modules. It was too unstructured and it had no subject-matter. This was why we turned to logic.

3 Why logic?

One of the main tasks of an academic writer is to give the reader reasons for believing certain things. Some courses on writing build on this idea. They encourage the student to think about the main kinds of reason one can use; for example one can quote a general rule, or one can point to analogies. Informal logic is sometimes used to provide a framework for this kind of analysis (for example [3]). We started with the same idea, but we reckoned at once that our mathematics students would need to dig deeper into the structure of mathematical arguments. They would need to be able to handle at least some of the natural deduction rules, though in English rather than in formal symbols.

Another line of thinking led us in the same direction. What famous mathematicians are also well known for their writing? Two names stand out at once: Blaise Pascal and Bertrand Russell. Today Bertrand Russell's literary achievements are in some danger of being forgotten, but he did win the Nobel Prize for Literature in 1950. His style is quaint for a modern reader—it reminds us strongly of (English) Georgian architecture—so we used him as a source of material for the students to precis or convert to modern style. (Actually there was another mathematician who won the Nobel Prize for Literature: the Spaniard Jose Echegaray in 1904 [4]. But at that time we hadn't heard of him, and little seems to survive of either his mathematics or his literature. Another famous literary mathematician was Lewis Carroll, but who knows his mathematics today?)

We noticed that both Pascal and Russell wrote about logic. Russell's logical contributions are widely known. Pascal is the main source of the notion of logical 'primitives' (otherwise known as non-logical constants), and both Peano and Tarski cited his logical work.

Pascal was particularly useful for us, because he thought a good deal about how to write well, both in mathematics and in other fields. (We relied mainly on his [5].)

Pascal believed that if you want to persuade someone else of something, you need to appeal to their mind and to their heart. You offer the evidence to their mind; to their heart you offer the incentive to care about the question. Pascal was a man of the seventeenth century, and today we have other ways of making essentially the same point. We say that a teacher needs to get on the same wavelength as the students, in order to put things in a way that the students can use and will want to use.

Pascal would have been puzzled by the way we teach logic today. He would have been particularly puzzled by the emphasis on elementary rules of inference, for example the rules of natural deduction. He supposed that the individual steps of logical reasoning are intuitively obvious and don't need to be taught. What does need to be taught, he claimed, is the overall strategy. His suggestions for a general strategy led us to the structure of our writing module.

According to Pascal, a person writing an exposition of mathematics needs to have two aims in parallel. The first aim is to make sure that the reader understands the words, and the second is to make sure that the reader believes the statements. The first involves identifying what the words refer to; the second involves arranging the steps in logical order. So we built the content of our course around two notions: identifying or defining things, and proving things.

We taught the module in second year. We reckoned that third year was too late. Some of our colleagues argued that second year was already too late, and maybe they were right. But our calculation was that the core first-year modules would have given the students raw material that we could call on for our purposes.

4 Identifying things

In a 'geometrical definition', says Pascal, we 'clearly identify something'. How do mathematicians clearly identify things?

Early in the course the students were set exercises where they had to describe some mathematical feature in words. For example they were shown a floral wallpaper pattern and told to imagine it spread across the whole plane. The question was to describe a feature of the pattern that is not also a feature of its mirror image, *without referring to the arrangement on the page*. So for example they weren't allowed to say 'at the top' or 'horizontal', but they could talk about walking along a line in the pattern and turning left. Some students had trouble giving a description that didn't also apply to the mirror image. For other wallpaper patterns, the students had to identify in words a mirror line or a centre of rotation. In the discussion groups the students found out whether other students could understand their identifications.

Besides these pictorial examples, the students also had to handle some more formal definitions. For example they were given the inductive definition of a set X of natural numbers:

(i) Every prime number is in X.
(ii) If m and n are numbers in X such that $m < n$ and m doesn't divide n, then mn is in X.
(iii) Nothing else is in X.

and they were required to show that 6, 15 and 90 are in X but 8 is not. The best way of proving that 8 is not in X is to find some property that prime numbers have, and that mn has whenever m, n are numbers as in (ii) that have it, and show that 8 doesn't have it. Lectures covered the general theory of inductive definitions and illustrated how to do exercises of this kind.

5 Proving things

Pascal's remarks on definition still make sense today, though we have to add to them. But his account of argument structure was way off mark. He thought that we reason like a Hilbert proof, where every statement is either an axiom or a logical consequence of earlier statements. In lectures we had some fun working through arguments that Pascal himself gave, for example in connection with his famous Triangle. A few of his statements are either assumed without proof or deduced from what precedes them. But what about the statement 'Let ω be a cell', or statements beginning 'Suppose ...', or places where he gives a diagram instead of a statement?

Pascal's main oversight was that he completely missed the notions of *making and discharging assumptions*. Statements of the form 'If A then B' are often proved by assuming A and deducing B. In fact this happens in such a regular way that if you can recognise how to paraphrase a theorem as 'If A then B', you can generally write down the first line of the proof. We gave the students exercises and exam questions where they had to do exactly this.

We took our examples from actual textbooks. For example how does a well-known combinatorics textbook begin the proof of the following theorem?

If $n \equiv 3 \pmod 6$, there exists a STS(N).

No prizes for guessing; it writes 'Suppose that $n \equiv 3 \pmod 6$ ...'. Other examples were subtler. For example how do geometers prove this theorem?

The number of elements in a minimal base for $L(u_1, \ldots, u_r)$ is the same for all bases.

In fact they write 'Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ be two minimal bases'. Why two? Most students saw at once that it won't do to write 'Let B be a minimal base' and aim to prove that B has the same number of elements. But many thought at first that we need to list all minimal bases. This led to some useful discussions in class.

One thing that emerged quickly from these examples was that you can often predict the first few lines of a proof, even when you haven't the faintest idea what the subject is. What is an STS(N), for example?

Richard Feynman ([1] p. 85f) has a beautiful account of how he gained a reputation for spotting mistakes in mathematicians' arguments in fields he knew nothing about.

> I had a scheme, which I still use today when somebody is explaining something that I'm trying to understand: I keep making up examples. For instance, the mathematicians would come in with a terrific theorem, and they're all excited. As they're telling me the conditions of the theorem, I construct something which fits all the conditions. You know, you have a set (one ball)—disjoint (two balls). Then the balls turn colors, grow hairs, or whatever, in my head as they put more conditions on. Finally they state the theorem, which is some dumb thing about the ball which isn't true for my hairy green ball thing, so I say, "False!". ... I guessed right most of the time because although the mathematicians thought their ... theorems were counterintuitive, they weren't really as difficult as they looked.

This was our excuse for studying some purely axiomatic arguments, using symbols with no defined meanings. We also looked at some examples of refuting symbolic implications by building counterexamples. Thus for instance:

> We consider the axioms:
>
> (a) Every bandersnatch snickers exactly one bandersnatch.
> (b) If x and y are bandersnatches, and x snickers y, then y snickers x.
>
> Show by proof or counterexample:
>
> (i) (a) implies that if the jubjub is a bandersnatch that snickers every bandersnatch, then every bandersnatch snickers the jubjub.
> (ii) (a) and (b) don't imply that if the number of bandersnatches is finite then it's even."

For questions of this sort we gave the students a framework for assigning meanings to symbols. This was the nearest we came to giving the students symbolic methods.

6 Assessing the course

To get meaningful evidence on the effectiveness of a course is a harder job than teaching the course, and it needs a completely different kind of expertise. That's true quite generally, but with this course there are added problems. Wilfrid taught it for only three experimental years, it changed during those years, for at least the first year the numbers were very small, and the skills being taught were quite subjective. Nobody has attempted a test of its efficiency, either on its own or in comparison with other courses elsewhere.

But at least we cleared another hurdle: we established that the module can sit alongside other mathematics modules without needing special concessions for the examining. The examiners' board at Queen Mary uses statistical tests to diagnose the general health of modules and their exams—for example it checks whether the marks on a module are seriously out of line with the marks of the same students on other modules. These tests gave our module a clean bill of health as a Mathematics module.

The student questionnaire responses were positive, and so were reactions of colleagues. Suffice it to say that the department has accepted the course as an important part of its portfolio, and it has now been taken over by Franco Vivaldi who is injecting new ideas.

There was a more conventional mathematical logic module taught in third year. We hoped that Logic I would serve as an introduction to that module. This didn't work out too well. Some students did move from our module to the third year logic, but they found it difficult to connect the material. Even when they had learned to use a natural deduction rule informally, they were baffled by the formal presentation of the same rule. Probably we could have overcome this problem by introducing more logical symbolism into the Mathematical Writing course, but this would hardly have matched the purpose of the course.

In fact only one mathematical logic text known to us contains material that we could use directly. This is the excellent *Introduction to Logic* of Patrick Suppes. His chapter 'Transition from formal to informal proofs' was right up our street. His example of a bogus argument ([7] p. 141):

Theorem. If $x + y = x + z$ then $y = z$.
Proof. Assume $x + y = x + z$. Put $x=0$. Then

$$y = 0 + y = 0 + z = z.$$

is a beautiful illustration of making an assumption and then failing to discharge it. Dan Velleman's book *How to Prove It* [8] also has good examples of bad arguments.

The fact remains that it's very difficult to teach logic students to do any more than answer exercises of sort that you train them in. Transfer of skills is still the Holy Grail. Today no logic teacher could express the sentiments of A. A. Luce with a straight face:

> In lecturing on Logic to University men and women, over and over again I have seen the same door swing open, the same step forward taken, the same marked development. The nerveless, juvenile letter written to the College Tutor twelve months previously, has now turned into a logical, well-knit communication, its subject-matter thought out, its phrasing clear and concise. Logic can do this for you. ([2] pp. vi, vii)

But it's a pleasure to be able to report that our Mathematical Writing course did precisely use logic to teach students to write logical, well-knit communications.

We plan to put some of our material into a textbook. We doubt that anyone else would want to teach exactly the module that we taught; but we can report what worked and what didn't, and we have a body of ideas and exercises to pass on.

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