M3/4PA45 Tilings and Patterns
Problems 5

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Problem 1 (assessed)

Consider the canonical projection method. Allow the window Ω to be any interval in W, not just the projection of a unit square. Show that the point set $\Pi_V(\mathbb{Z}^2 \cap V + \Omega)$ is the vertices of a tiling with at most three interval lengths.

Solution

Since Ω is bounded, this projection yields a discrete point set in V. We now consider the two points in $\mathbb{Z}^2 + t$ that project to either side of $\Pi_V(t)$ in V. We denote these points $t + a$ and $t - b$, where $a, b \in \mathbb{Z}^2$ correspond to steps of the staircase.

For simplicity, we continue the proof assuming that the window is the half-open interval $\Omega = [t, t + l) \subset W$. The argument for the remaining case in which $\Omega = (t, t + l] \subset W$ is analogous.

The subwindows corresponding to these two steps are the intervals $\Omega^a \subset \Omega$ and $\Omega^b \subset \Omega$ given by

$$\begin{aligned}\Omega^a &= \{p \in \Omega \mid \Pi_W(a) + p \in \Omega\} = [\Pi_W(t), \Pi_W(t + l - a)),\\ \Omega^b &= \{p \in \Omega \mid \Pi_W(b) + p \in \Omega\} = [\Pi_W(t - b), \Pi_W(t + l)).\end{aligned}$$

If $\Pi_W(t + l - a) = \Pi_W(t - b)$ then all steps in the staircase are accounted for, and there are only two types of steps in the staircase.

If $\Pi_W(t+l-a) \neq \Pi_W(t-b)$, apart from the subwindows for a and b discussed above, there is a remaining third subwindow in between. Consider a point $q \in \mathbb{R}^2$ whose projection to W lies in this central subwindow, i.e. $\Pi_W(q) \in [\Pi_W(t+l-a), \Pi_W(t-b))$. Then the projection to W of the point $q+a+b$ lies in the interval $[\Pi_W(t+l+b), \Pi_W(t+a))$ which is a subinterval of Ω. We finally claim that if $q \in \mathbb{Z}^2$ projects to the central subwindow, the next (right) step in the staircase is $a + b$. Namely, if there are points whose projection to V lies between $\Pi_V(q)$ and $\Pi_V(q + a + b)$, then by periodicity of the lattice $t + a$ and $t - b$ would not project to the closest points to t, contradicting the definition of a and b.

Hence, there are three types of staircase steps: a, b and $a + b$. The corresponding subwindows are $\Omega^a = [\Pi_W(t), \Pi_W(t+l-a))$, $\Omega^{a+b} = [\Pi_W(t+l-a), \Pi_W(t-b))$ and $\Omega^b = [\Pi_W(t-b), \Pi_W(t+l))$.

In the case of the other choice of the half-open window interval $\Omega = (\Pi_W(t), \Pi_W(t + l)]$, one finds $\Omega^a = (\Pi_W(t), \Pi_W(t+l-a)]$, $\Omega^{a+b} = (\Pi_W(t+l-a), \Pi_W(t-b)]$ and $\Omega^b = (\Pi_W(t-b), \Pi_Y(t+l)]$.

Problem 2

Prove that the symbolic sequence of a Canonical projection tiling is a minimal sequence.

Solution

In Theorem 11.7 of the course the proof shows every patch of a canonical projection tiling has a window. Every patch occurs precisely when a point lands in this window. To show that the tiling is minimal therefore we need to show that the tiling cannot stay outside a window for arbitarily many steps. Consider a patch P and its projection to W. This is a canonical projection point set with window given by the length of the patch. By Problem 1 therefore there are at most 3 step lengths. As the patch gets longer these steps get smaller going to zero and the patch length goes to infinity. Thus for any window of positive length a patch of sufficient length must contain points in the window. The sequence is therefore minimal.

Problem 3

3.1

Find all two letter substitution rules with scaling $2+\sqrt{3}$.

Solution

As we want to find two letter substitutions the incidence matrix must be a two by two matrix. Consider the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. As all four entries are positive the leading eigenvalue, that gives the scaling is:

$$\frac{1}{2}\left(a+d+\sqrt{a^2+4bc-2ad+d^2}\right)$$

We therefore have $a+d=4$, replacing d by $4-a$ we have:

$$2+\sqrt{4-4a+a^2+bc}$$

We therefore have $1+a^2+bc-4a=0$. As a has a maximum value of 4 there are therefore only a finite number of possible positive values for b, c for each value of a. First consider $a=4$, thus $d=0$ and $1+16+b+c-16=0$, so no values of b and c are possible. Now if $a=4$, $d=0$ and $1+9+bc-12=0$, so $bc=2$. Thus we have $b=2, c=1$ or $b=1, c=2$. We get the following matrices:

a	Matrices
3	$\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$
2	$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$
1	$\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

To find the substitution rules for each matrix simply give all possible orders of the letters that give the right incidence matrix. For example consider the matrix: $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$. This has substitution rules:

$a \to aabbb$	$b \to ab$	$a \to aabbb$	$b \to ba$
$a \to ababb$	$b \to ab$	$a \to ababb$	$b \to ba$
$a \to abbab$	$b \to ab$	$a \to abbab$	$b \to ba$
$a \to abbba$	$b \to ab$	$a \to abbba$	$b \to ba$
$a \to baabb$	$b \to ab$	$a \to baabb$	$b \to ba$
$a \to babab$	$b \to ab$	$a \to babab$	$b \to ba$
$a \to babba$	$b \to ab$	$a \to babba$	$b \to ba$
$a \to bbaab$	$b \to ab$	$a \to bbaab$	$b \to ba$
$a \to bbaba$	$b \to ab$	$a \to bbaba$	$b \to ba$
$a \to bbbaa$	$b \to ab$	$a \to bbbaa$	$b \to ba$

3.2

List the invertible substitutions.

Solution

In the list above:

$a \to ababb$	$b \to ab$	$a \to abbab$	$b \to ab$
$a \to babab$	$b \to ab$	$a \to babab$	$b \to ba$
$a \to babba$	$b \to ba$	$a \to bbaba$	$b \to ba$

3.3

How many local isomorphism class of tilings are generated by these substitution rules?

Solution

The invertible substitution rules above all generate the same tiling. In fact each matrix above gives a single LI class for its invertible substitutions. The different matrices cannot give the same class as they have different eigenvectors.

To show that the substitution rules generate the same LI class of tiling consider a substitution rule where both replacing words start with the same letter. Swapping the letter to the end will not change the local isomorphism class as every letter at the beginning of each expanded word is moved to the end of the word before.

ab with *ba*. We established in the course that this did not change the LI class. Starting with $a \to babab$, $b \to ba$ goes to $a \to ababb$, $b \to ab$ to $a \to babba$, $b \to ba$ to $a \to abbab$, $b \to ab$ to $a \to bbaba$, $b \to ba$ to $a \to babab$, $b \to ab$.

Problem 4

Consider the invertible substitution rules from Problem sheet 4. Prove using a division of the window into intervals that they generate Canonical projection tilings.